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Kernels Associated to General Elliptic Problems*

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Let $(A, (B_j)_{0 \leqslant j \leqslant m-1})$ be a regular elliptic boundary value problem in R_+^{n+1} (see definition at Section 3).

To $(A, (B_j)_{0 \leqslant j \leqslant m-1})$ we associate m kernels $(K_j)_{0 \leqslant j \leqslant m-1}$ which are temperated distributions in R^{n+1} verifying the following properties:

a) K_j is a C^∞ function in $R^{n+1} - \{0\}$ (in fact, it is analytic but we only prove differentiability);

b) if $m_j < n$, then K_j is a homogeneous distribution of degree $m_j - n$; if $m_j \geqslant n$, then $K_j = G_j + P_j \cdot \ln(x_1{}^2 + \cdots + x_n{}^2 + t^2)^{1/2}$, where G_j is a homogeneous function of degree $m_j - n$ and P_j is a homogeneous polynomial of degree $m_j - n$;

c) each K_j verifies the boundary value problem

$$\begin{cases} AK_j = 0 & \text{in} \quad R_+^{n+1} \\ B_l K_j \,|_{R^{n-1}} = \delta_{l,j} \cdot \delta, & 0 \leqslant l \leqslant m-1, \end{cases}$$

where $\delta_{l,j}$ is the Kronecker symbol, δ is the Dirac measure and

$$B_l K_j \,|_{R^{n-1}} = \lim_{t \to 0+} B_l K_j(x, t),$$

the limit being taken in $\mathcal{S}'(R^n)$ the space of temperated distributions in R^n.

Actually, the kernels $(K_j)_{0 \leqslant j \leqslant m-1}$ are elementary solutions of the given elliptic boundary value problem in the following sense. If g_j, $0 \leqslant j \leqslant m-1$, are given functions, say in $\mathcal{S}(R^n)$, then

$$u(x, t) = \sum_{k=0}^{m-1} \int_{R^n} K_j(x-y, t)\, g_j(y)\, dy$$

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is a solution of the boundary value problem

$$\begin{cases} Au = 0 & \text{in} \quad R_+^{n+1} \\ B_j u_{R^n} = g_j\,, & 0 \leqslant j \leqslant m-1. \end{cases}$$

Kernels similar to the ones here considered, were used by Agmon–Douglis–Nirenberg [*1*] and by Browder [*3*] in connection with their work on *a priori* estimates for elliptic problems.

In this paper we present a new and general method to obtain such kernels. Our method is based on the study of distributions defined by the *finite part* of divergent integrals and their Fourier transforms. Homogeneous functions of degree $-m$, where m is an integer $\geqslant 0$, are known ([*4*], [*6*]) to define temperated distributions. When $m \geqslant n$, such a distribution is defined by the finite part of a divergent integral (see (1.3)).

When applied to the Dirichlet problem in a half plane, our method gives, in particular, the Poisson kernel as we have shown in [*2'*].

Section 1 is devoted to a brief discussion of homogeneous distributions and their Fourier transform. In Section 2 we prove properties of differentiability and homogeneity of $\mathcal{F}U$ the Fourier transform of a homogeneous distribution U. The results of Sections 1 and 2 are used in Sections 3, 4 and 5 to obtain the kernels K_j, $0 \leqslant j \leqslant m-1$, verifying the conditions a), b) and c) above.

1. On Some Pseudo-Functions and Their Fourier Transform

Let R^n be the euclidean n-dimensional space. Denote by $x = (x_1, ..., x_n)$ a variable element in R^n, by

$$|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$$

its distance to the origin. If $p = (p_1, ..., p_n)$ is a n-tuple of integers $p_i \geqslant 0$, let

$$\frac{\partial^p}{\partial x^p} = \frac{\partial^{p_1 + \cdots + p_n}}{\partial x_1^{p_1} \cdots \partial x_n^{p_n}}$$

be a partial derivative of order $|p| = p_1 + \cdots + p_n$. We also set $D_j = 1/i\, \partial/\partial x_j$, where $i = \sqrt{-1}$, and $D^p = D_1^{p_1} \cdots D_n^{p_n}$.

Let $\mathcal{S}(R^n)$ be the space of C^∞ functions in R^n rapidly decreasing at infinity, i.e., such that

$$\lim_{|x| \to \infty} |x^k D^p \phi(x)| = 0,$$

for all p and k. In $\mathcal{S}(R^n)$ we define a locally convex topology given by the family of semi-norms

$$\gamma_{p,k}(\phi) = \sup_{x \in R^n} | x^k D^p \phi(x)|.$$

Equipped with this topology, $\mathcal{S}(R^n)$ is a locally convex, metrisable and complete space.

The dual of $\mathcal{S}(R^n)$ is the space of *temperated distributions* in R^n which is denoted by $\mathcal{S}'(R^n)$.

We want to study some particular, but important, temperated distributions which are related in a natural way to regular elliptic boundary problems.

Let $U(x)$ be a homogeneous function of degree $-m$ in R^n, where m is an integer $\geqslant 0$. If we set $r = | x |$, we can write

$$U(x) = r^{-m} f(\omega) \tag{1.1}$$

with $\omega = (\omega_1, ..., \omega_n) \in \Omega$, where Ω denotes the unit sphere in R^n. We suppose always that $f(\omega)$ is a C^∞ function on Ω.

The function $U(x)$ defines a temperated distribution in R^n. We have two cases to consider, namely, $m < n$ or $m \geqslant n$.

If $m < n$, then $U(x)$ is a locally integrable function and, as it is well known, locally integrable functions define temperated distributions ([*4*], [*6*]).

If $m \geqslant n$, we have to use the notion of *finite part* of a divergent integral ([*4*], [*6*]). We proceed as follows. Let ϕ be an element of $\mathcal{S}(R^n)$ and let

$$u_\phi(r) = \int_\Omega f(\omega) \phi(r\omega) \, d\omega. \tag{1.2}$$

Define next

$$\langle U, \phi \rangle = Pf \int_0^\infty r^{-m+n+1} u_\phi(r) \, dr, \qquad \forall \phi \in \mathcal{S}(R^n),[1] \tag{1.3}$$

where the symbol Pf means that we are considering the finite part of the integral. It is easy to see that (1.3) defines a temperated distribution in R^n ([*4*], [*6*]). Also, if $m < n$, the integral which appears in (1.3) converges, so that, the same formula without the Pf defines U as a temperated distribution in R^n.

[1] Distributions defined by means of finite parts of divergent integrals are called *pseudo-functions*.

If $m = n$ the finite part of integral (1.3) is equal to:

$$\langle U, \phi\rangle = \int_0^1 r^{-1}\{u_\phi(r) - 1\} + \int_1^\infty r^{-1} \cdot u_\phi(r)\, dr. \tag{1.4}$$

When $m > n$, the finite part of the integral in (1.3) is shown to be equal to

$$\langle U, \phi\rangle = \int_0^1 r^{-m+n-1} \left\{u_\phi(r) - \sum_{s=0}^{m-n} \frac{1}{s!} \frac{\partial^s u_\phi(0)}{\partial r^s} r^s\right\} dr$$

$$+ \int_1^\infty r^{-m+n-1} \left\{u_\phi(r) - \sum_{s=0}^{m-n-1} \frac{1}{s!} \frac{\partial^s u_\phi(0)}{\partial r^s} r^s\right\} dr, \tag{1.5}$$

where these two integrals are absolutely convergent.

Since U is a temperated distribution, its Fourier transform $\mathcal{F}U$ is also a temperated distribution. In many applications (see [2]) it is useful to have a formula representing $\mathcal{F}U$ by means of uniformly convergent integrals, whenever such a formula exists.

PROPOSITION 1. *If $m > n$, the following representation formula holds*:

$$\mathcal{F}U(\xi) = \int_{R^n} \mathfrak{X}(x) \cdot U(x)\Big(e^{-i\langle x,\xi\rangle} - \sum_{s=0}^{m-n} \frac{1}{s!}(-i\langle x, \xi\rangle)^s\Big)\, dx$$

$$+ \int_{R^n} (1 - \mathfrak{X}(x)) \cdot U(x)\Big(e^{-i\langle x,\xi\rangle} - \sum_{s=0}^{m-n-1} \frac{1}{s!}(-i\langle x, \xi\rangle)^s\Big)\, dx, \tag{1.6}$$

where $\mathfrak{X}(x)$ is the characteristic function of the unit ball and the two integrals are absolutely convergent.

Proof. Let U_1(resp. U_2) be the temperated distribution defined by the first (resp. second) integral in (1.5). We have by the definition of Fourier transform of a distribution:

$$\langle \mathcal{F}U_1, \phi\rangle = \langle U_1, \hat{\phi}\rangle\ ^2$$

$$= \int_0^1 r^{-m+n-1} \left\{u_{\hat{\phi}}(r) - \sum_{s=0}^{m-n} \frac{1}{s!} \frac{\partial^s u_{\hat{\phi}}(0)}{\partial r^s} \cdot r^s\right\} dr. \tag{1.7}$$

But,

$$u_{\hat{\phi}}(r) = \int_\Omega \int_{R^n} e^{-i\langle \xi, r\omega\rangle} f(\omega)\, \phi(\xi)\, d\xi\, d\omega$$

[2] We set $\hat{\phi} = \mathcal{F}\phi$, $\forall \phi \in \mathcal{S}(R^n)$.

and

$$\frac{\partial^s u_\phi(0)}{\partial r^s} = \int_\Omega \int_{R^n} (-i\langle \xi, r\omega\rangle)^s f(\omega)\, \phi(\xi)\, d\xi\, d\omega.$$

Replacing in (1.7) we get:

$$\langle \mathcal{F} U_1, \phi\rangle = \int_0^1 r^{-m+n-1} \left\{ \int_\Omega \int_{R^n} e^{-i\langle \xi, r\omega\rangle} f(\omega)\, \phi(\xi)\, d\xi\, d\omega \right.$$
$$\left. - \sum_{s=0}^{m-n} \frac{1}{s!} r^s \int_\Omega \int_{R^n} (-i\langle \xi, r\omega\rangle)^s f(\omega)\, \phi(\xi)\, d\xi\, d\omega \right\} dr.$$

It is easily seen that these integrals are absolutely convergent. By changing the order of integration and by changing variables we get:

$$\langle \mathcal{F} U_1, \phi\rangle$$
$$= \int_{R^n} \left\{ \int_0^1 \int_\Omega r^{-m+n-1} f(\omega) \left(e^{-i\langle \xi, r\omega\rangle} - \sum_{s=0}^{m-n} \frac{1}{s!} (-i\langle \xi, \omega\rangle)^s r^s \right) dr\, d\omega \right\} \phi(\xi)\, d\xi$$
$$= \left\langle \int_{R^n} \mathcal{X}(x)\, U(x) \left(e^{-i\langle x, \xi\rangle} - \sum_{s=0}^{m-n} \frac{1}{s!} (-i\langle x, \xi\rangle)^s \right) dx, \phi(\xi) \right\rangle, \quad \forall \phi \in \mathcal{S}(R^n). \tag{1.8}$$

In a similar way, we get:

$$\langle \mathcal{F} U_2, \phi\rangle = \langle U_2, \hat{\phi}\rangle = \int_{R^n} \left\{ \int_1^\infty \int_\Omega r^{-m+n-1} f(\omega) \left(e^{-i\langle \xi, r\omega\rangle} \right.\right.$$
$$\left.\left. - \sum_{s=0}^{m-n-1} \frac{1}{s!} (-i\langle \xi, \omega\rangle)^s r^s \right) dr\, d\omega \right\} \phi(\xi)\, d\xi, \tag{1.9}$$

this integral being absolutely convergent. In fact, we have the estimates

$$\left| e^{-i\langle \xi, r\omega\rangle} - \sum_{s=0}^{m-n-1} \frac{1}{s!} (-i\langle \xi, \omega\rangle)^s r^s \right| \leqslant 1 + \sum_{s=0}^{m-n-1} \frac{|\langle \xi, r\omega\rangle|^s}{s!}$$
$$\leqslant c \cdot (1 + |\langle \xi, r\omega\rangle|^{m-n-1}).$$

On the other hand, the integral

$$\int_1^\infty \int_\Omega r^{-m+n-1} (1 + |\langle \xi, r\omega\rangle|^{m-n-1})\, dr\, d\omega$$

converges. These two facts imply the absolute convergence of (1.9).

Next, by changing variables in (1.9) we obtain:

$$\langle \mathcal{F} U_2, \phi\rangle = \Big\langle \int_{R^n} (1 - \mathfrak{X}(x))\, U(x) \Big(e^{-i\langle x,\xi\rangle} - \sum_{s=0}^{m-n-1} \frac{1}{s!} (-i\langle x, \xi\rangle)^s\Big)\, dx, \phi(\xi)\Big\rangle, \qquad \forall \phi \in \mathcal{S}(R^n). \tag{1.10}$$

The two formulas (1.8) and (1.10) imply (1.6). Q.E.D.

Now, let us examine the representation formula when $m \leqslant n$. Let $\mathfrak{X}(x)$ be the characteristic function of the unit ball and let

$$Z(x) = (1 - \mathfrak{X}(x))\, U(x).$$

Then, $Z(x)$ defines a temperated distribution and we have:

$$\langle Z, \phi\rangle = \int_1^\infty r^{-1} u_\phi(r)\, dr.$$

Proceeding as we did in Proposition 1, we get the following representation formula for $\mathcal{F} U$, when $m = n$

$$(\mathcal{F} U)(\xi) = \int_{R^n} \mathfrak{X}(x)\, U(x)(e^{-i\langle x,\xi\rangle} - 1)\, dx + \mathcal{F} Z. \tag{1.11}$$

When $m < n$, we get:

$$(\mathcal{F} U)(\xi) = \int_{R^n} \mathfrak{X}(x)\, U(x)\, e^{-i\langle x,\xi\rangle}\, dx + \mathcal{F} Z. \tag{1.12}$$

In general, we cannot represent $\mathcal{F} Z$ by means of an absolutely convergent integral.

2. Differentiability and Homogeneity of $\mathcal{F} U$

First, we state a result concerning the differentiability of $\mathcal{F} U$.

THEOREM 1. *Let $U(x)$ be the homogeneous function considered in the previous section. Its Fourier transform $\mathcal{F} U$ is a C^∞ function in $R^n - \{0\}$.*

Proof. Let α be a function of $\mathcal{D}(R^n)$ such that $\alpha = 1$ in a neighborhood of 0. Write

$$U = \alpha U + (1 - \alpha) U$$

and

$$\mathcal{F}U = \mathcal{F}(\alpha U) + \mathcal{F}((1-\alpha)U).$$

Since αU is a distribution with compact support, then $\mathcal{F}(\alpha U)$ is an analytic function in the whole R^n ([5]).

Next, consider the distribution $\mathcal{F}((1-\alpha)U)$. We have to distinguish two cases, namely: a) $m \leqslant n$; b) $m > n$.

Case a). Since U is homogeneous of degree $-m$, the derivative

$$D_j^{n-m+1}((1-\alpha)U)$$

is an integrable function in R^n. Thus, its Fourier transform

$$\mathcal{F}(D_j^{n-m+1}((1-\alpha)U)) = x_j^{n-m+1}\mathcal{F}((1-\alpha)U)$$

is a bounded function in R^n, for each $1 \leqslant j \leqslant n$. This result implies that

$$|x|^{n-m+1}\mathcal{F}((1-\alpha)\,U)$$

is a bounded function in R^n, thus, $\mathcal{F}((1-\alpha)U)$ is a function in $R^n - \{0\}$.

A similar argument shows also that $D^r\mathcal{F}((1-\alpha)U)$ is a function in $R^n - \{0\}$, $\forall r$. Consequently, $\mathcal{F}((1-\alpha)U)$ is a C^∞ function in $R^n - \{0\}$.

Case b). In this case, $(1-\alpha)U$ is integrable in R^n. In fact, we have:

$$\int_{R^n} |(1-\alpha)U|\, dx = \int_a^\infty \int_\Omega r^{-m}|f(\omega)|\, r^{n-1}\, dr\, d\omega \leqslant c \cdot \int_a^\infty r^{-m+n-1}\, dr$$

and the last integral converges, because $m > n$.

Hence, $\mathcal{F}((1-\alpha)U)$ is a bounded function in R^n. In the same way, we conclude that $D^r\mathcal{F}((1-\alpha)U)$ is also a function in R^n, $\forall r$. Thus, $\mathcal{F}((1-\alpha)U)$ is a C^∞ function in R^n.

Summing up these results, we conclude that, in each case $m \lesseqgtr n$, the temperated distribution $\mathcal{F}U$ is a C^∞ function in $R^n - \{0\}$, Q.E.D.

We recall the

DEFINITION. *A distribution $T \in \mathcal{D}'(R^n)$ is homogeneous of degree k if*

$$\langle T, \phi(c^{-1}x)\rangle = c^{k+n}\langle T, \phi\rangle,$$

$\forall c > 0$ *and* $\forall \phi \in \mathcal{D}(R^n)$.

When T is a function, this definition coincides with the classical definition of a homogeneous function.

THEOREM 2. *If $m < n$, then $\mathcal{F}U$ is a homogeneous distribution of degree $m - n$, defined by a function which is infinitely differentiable in $R^n - \{0\}$.*

Proof. It is easy to verify, by direct inspection, that $\mathcal{F}U$ is a homogeneous *distribution* of degree $m - n$.

Furthermore, Theorem 1 tells us that $\mathcal{F}U$ coincides with a C^∞ function in $R^n - \{0\}$. Thus,

$$\mathcal{F}U = F + T,$$

where F is a C^∞ function outside the origin and T is a distribution with support at the origin

$$T = \sum c_\alpha \, \delta^{(\alpha)}.$$

It follows that T must be a homogeneous distribution of degree $m - n$, because $\mathcal{F}U$ and T have that degree of homogeneity. But, then the relations

$$\langle T, \phi(c^{-1}x)\rangle = \sum c_\alpha \cdot c^{-|\alpha|} \, D^\alpha \phi(0)$$

and

$$\langle T, \phi(c^{-1}x)\rangle = c^m \sum c_\alpha \cdot c^{-|\alpha|} \, D^\alpha \phi(0)$$

being true for all $c > 0$, they imply $c_\alpha = 0, \forall \alpha$, i.e., $T = 0$. Thus, $\mathcal{F}U = F$ a C^∞ function in $R^n - \{0\}$, Q.E.D.

THEOREM 3. *If $m \geqslant n$, then $\mathcal{F}U$ is a locally function in R^n, such that*

$$\mathcal{F}U = H + P \cdot \ln|x|, \tag{2.1}$$

where H is a homogeneous function of degree $m - n$, infinitely differentiable in $R^n - \{0\}$ and where P is a homogeneous polynomial of degree $m - n$.

Proof. 1. Suppose $m > n$. Let $\phi \in \mathcal{S}(R^n)$ and write $\psi(x) = \phi(c^{-1}x)$. We have,

$$\langle \mathcal{F}U, \psi\rangle = \langle U, \mathcal{F}\psi\rangle = \int_0^\infty \mathcal{X}(r) \cdot r^{-m+n-1} \left\{u_{\hat{\psi}}(r) - \sum_{s=0}^{m-n} \frac{1}{s!} \frac{\partial^s u_{\hat{\psi}}(0)}{\partial r^s} r^s\right\} dr$$

$$+ \int_0^\infty (1 - \mathcal{X}(r)) \cdot r^{-m+n-1} \left\{u_{\hat{\psi}}(r) - \sum_{s=0}^{m-n-1} \frac{1}{s!} \frac{\partial^s u_{\hat{\psi}}(0)}{\partial r^s} r^s\right\} dr, \tag{2.2}$$

where $\mathfrak{X}(r)$ is a C^∞ function of r, $=1$ when, $0 \leqslant r \leqslant 1$, and, $=0$ when $r \geqslant 2$.

Next, recalling the definition of u_{ψ}, we have the following relations:

$$u_{\psi}(r) = c^n u_{\phi}(cr)$$

and

$$\frac{\partial^s u_{\psi}(0)}{\partial r^s} = c^{n+s} \frac{\partial^s u_{\phi}(0)}{\partial r^s}.$$

Replacing the relations in (2.2), we get:

$$\langle \mathcal{F} U, \psi \rangle = \langle U, \mathcal{F}\psi \rangle = c^n \int_0^\infty \mathfrak{X}(r)\, r^{-m+n-1} \left\{ u_{\phi}(cr) - \sum_{s=0}^{m-n} \frac{1}{s!} \frac{\partial^s u_{\phi}(0)}{\partial r^s} (cr)^s \right\} dr$$

$$+ c^n \int_0^\infty (1 - \mathfrak{X}(r))\, r^{-m+n-1} \left\{ u_{\phi}(cr) - \sum_{s=0}^{m-n-1} \frac{1}{s!} \frac{\partial^s u_{\phi}(0)}{\partial r^s} (cr)^s \right\} dr.$$

By changing variables we get:

$$\begin{aligned} &\langle \mathcal{F} U, \phi(c^{-1}x) \rangle \\ &= c^m \int_0^\infty \mathfrak{X}(c^{-1}r)\, r^{-m+n-1} \left\{ u_{\phi}(r) - \sum_{s=0}^{m-n} \frac{1}{s!} \frac{\partial^s u_{\phi}(0)}{\partial r^s} r^s \right\} dr \\ &\quad + c^m \int_0^\infty (1 - \mathfrak{X}(c^{-1}r))\, r^{-m+n-1} \left\{ u_{\phi}(r) - \sum_{s=0}^{m-n-1} \frac{1}{s!} \frac{\partial^s u_{\phi}(0)}{\partial r^s} r^s \right\} dr. \end{aligned} \tag{2.3}$$

From (2.3), it follows that

$$\langle \mathcal{F} U, \phi(c^{-1}x) \rangle - c^m \langle \mathcal{F} U, \phi \rangle = \frac{c^m}{(m-n)!} \int_0^\infty \frac{\mathfrak{X}(r) - \mathfrak{X}(c^{-1}r)}{r} \frac{\partial^{m-n} u_{\phi}(0)}{\partial r^{m-n}}\, dr. \tag{2.4}$$

On the other hand, recalling the definition of u_{ϕ}, it is easy to see that

$$\begin{aligned} \frac{\partial^{m-n} u_{\phi}(0)}{\partial r^{m-n}} &= \int_\Omega f(\omega) \left\{ \int_{R^n} (i\langle x, \omega \rangle)^{m-n} \phi(x)\, dx \right\} d\omega \\ &= \left\langle \int_\Omega f(\omega)(i\langle x, \omega \rangle)^{m-n}\, d\omega, \phi \right\rangle. \end{aligned}$$

Hence, replacing in (2.4), we can write:

$$\begin{aligned} &\langle \mathcal{F} U, \phi(c^{-1}x) \rangle - c^m \langle \mathcal{F} U, \phi \rangle \\ &= c^m \left\langle \int_0^\infty \int_\Omega \frac{\mathfrak{X}(r) - \mathfrak{X}(c^{-1}r)}{r} f(\omega) \frac{(i\langle x, \omega \rangle)^{m-n}}{(m-n)!}\, dr\, d\omega, \phi \right\rangle. \end{aligned} \tag{2.5}$$

Now, the integral appearing inside the brackets is, as one can easily see, a polynomial $Q(x)$, homogeneous of degree $m - n$, whose coefficients are:

$$a_\alpha(c) = \frac{i^{m-n}}{(m-n)!} \int_0^\infty \int_\Omega \frac{\mathfrak{X}(r) - \mathfrak{X}(c^{-1}r)}{r} f(\omega)\, w^\alpha\, dr\, d\omega,$$

with $\alpha = (\alpha_1, \ldots, a_n)$ and $|\alpha| \leqslant m - n$.

Next, one can check that $a_\alpha(c)$ is a continuously differentiable function of $c > 0$ and that $a_\alpha(1) = 0$ and $a_\alpha(c_1 \cdot c_2) = a_\alpha(c_1) + a_\alpha(c_2)$. Thus

$$a_\alpha(c) = b_\alpha \cdot \ln c$$

where b_α is a constant. It follows that $Q(x) = P(x) \cdot \ln c$.

Consider now the following distribution

$$H = \mathcal{F}U - P(X) \cdot \ln|x|.$$

It is simple to verify, using (2.5), that H is a homogeneous distribution of degree $m - n$. Since, by Theorem 1, $\mathcal{F}U$ is a C^∞ function in $R^n - \{0\}$, then H is also an infinitely differentiable function off the origin.

It follows that H should be the sum of a C^∞ function in $R^n - \{0\}$ plus a homogeneous distribution with support at the origin. With the same argument we already used in the proof of Theorem 2, we can conclude that this distribution must be zero.

Thus,

$$\mathcal{F}U = H + P \cdot \ln|x|$$

where H is a homogeneous function of degree $m - n$, C^∞ in $R^n - \{0\}$, and P a homogeneous polynomial.

2. The case $m = n$ is even simpler. Thus, Theorem 3 is proved.

3. An Application to Regular Elliptic Boundary Valued Problem

Denote by R_n^{n+1} the half space

$$\{(x, t) \in R^{n+1} : t > 0\}$$

and by $\overline{R_+^{n+1}}$ the closure of R_+^{n+1} in R^{n+1}. Let

$$|(x, t)| = (x_1^2 + \cdots + x_n^2 + t^2)^{1/2}.$$

If $p = (p_1, \ldots, p_n, p_{n+1})$, let

$$D^p = D_1^{p_1} \cdots D_n^{p_n} \cdot D_t^{p_{n+1}},$$

where

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j}, \qquad 1 \leqslant j \leqslant n, \qquad \text{and} \qquad D_t = \frac{1}{i}\frac{\partial}{\partial t}.$$

If $f \in \mathcal{S}(R_+^{n+1})$, its partial Fourier transform with respect to the variable $x = (x_1, \ldots, x_n)$ is defined by

$$(\mathcal{F}f)(\xi, t) = \int_{R^n} e^{-i\langle x,\xi\rangle} f(x, t)\, dx$$

where $\xi = (\xi_1, \ldots, \xi_n)$ and $\langle x, \xi\rangle = x_1\xi_1 + \cdots + x_n\xi_n$.

If $g \in \mathcal{S}(R_+^{n+1}$, its inverse partial Fourier transform is defined by

$$(\mathcal{F}^{-1}g)(x, t) = (2\pi)^{-n} \int_{R^n} e^{i\langle x,\xi\rangle} g(\xi, t)\, d\xi.$$

The partial Fourier transform (resp. inverse partial Fourier transform), in $\mathcal{S}'(R_+^{n+1})$ is defined by duality

$$\langle \mathcal{F}T, \phi\rangle = \langle T, \mathcal{F}\phi\rangle$$

(resp. $\langle \mathcal{F}^{-1}T, \phi\rangle = \langle T, \mathcal{F}^{-1}\phi\rangle$), $\forall \phi \in \mathcal{S}(R^{n+1})$.

We also denote by $\hat{f}$ and $\hat{T}$ the partial Fourier transforms $\mathcal{F}f$ and $\mathcal{F}T$. Let

$$A = \sum_{|p|=2m} a_p D^p$$

be a partial differential operator with constant coefficients $a_p \in C$, homogeneous of degree $2m$, and *elliptic*, i.e., there is a constant $C_0 > 0$ such that

$$|a(\xi, \lambda)| \leqslant C_0 |(\xi, \lambda)|^{2m}, \qquad \forall (\xi, \lambda) \in R^{n+1},$$

where

$$a(\xi, \lambda) = \sum_{|p|=2m} a_p \xi_1^{p_1} \cdots \xi_n^{p_n} \cdot \lambda^{p_{n+1}}$$

is the *characteristic polynomial* of A.

Let

$$B_j = \sum_{|q|=m_j} b_{j,q} D^q, \qquad 0 \leqslant j \leqslant m-1,$$

be a partial differential operator, with constant coefficients, homogeneous of degree $m_j < 2m$. Let $b_j(\xi, \lambda)$ be the characteristic polynomial of B_j.

DEFINITION. *We shall say that the system $(A(B_j)_{0 \leqslant j \leqslant m-1})$ defines a regular elliptic problem in R_+^{n+1} if the following conditions are verified:*

1) *A is an elliptic operator;*

2) *if $\xi = (\xi_1, ..., \xi_n) \in R^n$, then $a(\xi, \lambda) = 0$ has exactly m roots (counting multiplicities) with positive imaginary part;*

3) *let C be a smooth Jordan curve, containing all the roots of $a(\xi, \lambda) = 0$ in the upper half complex plane, whenever $|\xi| = 1$. Let C_ξ be the curve*

$$C_\xi = |\xi| C = \{\lambda = |\xi| \mu : \mu \in C\}, \qquad \xi \in R^n,$$

and let

$$p_{j,k}(\xi) = \int_{C_\xi} \frac{\lambda^k b_j(\xi, \lambda)}{a(\xi, \lambda)} \, d\lambda \tag{3.1}$$

with $0 \leqslant j \leqslant m-1$ and $0 \leqslant k \leqslant m-1$. Then, the matrix $\| p_{j,k}(\xi) \|$ is supposed to be nonsingular.

Given a regular elliptic problem $(A, (B_j)_{0 \leqslant j \leqslant m-1})$ in R_+^{n+1}, we want to find m temperated distributions $K_j(x, t)$, $0 \leqslant j \leqslant m-1$, defined on R_+^{n+1} and verifying the boundary valued problem

$$\begin{cases} AK_j = 0 & \text{in } \ R_+^{n+1} \\ B_l K_j |_{R^n} = \delta_{l,j} \cdot \delta, & 0 \leqslant l \leqslant m-1, \end{cases} \tag{3.2}$$

where $\delta_{l,j}$ is the Kronecker symbol, δ is the Dirac measure in R^n and

$$B_l K_j |_{R^n} = \lim_{t \to 0+} B_l K_j(x_1, ..., x_n, t),$$

the limit being taken in $\mathcal{S}'(R^n)$.

Taking partial Fourier transform with respect to $x = (x_1, ..., x_n)$, equations (3.2) change into equations

$$\begin{cases} \left[a\left(\xi, \dfrac{1}{i} \dfrac{\partial}{\partial t}\right) \hat{K}_j \right](\xi, t) = 0 \\ \left[b_l\left(\xi, \dfrac{1}{i} \dfrac{\partial}{\partial t}\right) \hat{K}_j \right](\xi, 0) = \delta_{l,j}, \qquad 0 \leqslant l \leqslant m-1. \end{cases} \tag{3.3}$$

Now, let

$$s_j(\lambda) = \sum_{k=0}^{m-1} C_{k,j} \lambda^k, \qquad 0 \leqslant j \leqslant m-1,$$

be a polynomial of degree $\leqslant m - 1$, with coefficients $C_{k,j}$ to be determined.

For each j, the function

$$U_j(\xi, t) = \int_{C_\xi} \frac{s_j(\lambda)\, e^{i\lambda t}}{a(\xi, \lambda)}\, d\lambda,$$

verifies trivially the equation

$$a\left(\xi, \frac{1}{i}\frac{\partial}{\partial t}\right) U_j = 0.$$

Next, we determine the coefficients $C_{k,j}$, $0 \leqslant k \leqslant m - 1$, so that the initial conditions (3.3) hold true. This is possible because of assumption 3) in the definition of a regular elliptic problem. In fact, we have

$$\left[b_l\left(\xi, \frac{1}{i}\frac{\partial}{\partial t}\right) U_j\right](\xi, 0) = \sum_{k=0}^{m-1} C_{k,j} \int_{C_\xi} \frac{b_l(\xi, \lambda)\, \lambda^k}{a(\xi, \lambda)}\, d\lambda = \delta_{l,j}, \qquad 0 \leqslant l \leqslant m - 1. \tag{3.4}$$

But, by assumption,

$$\det \| p_{j,k}(\xi) \| \neq 0.$$

Thus the linear system (3.4), in the unknowns $(C_{k,j})_{0 \leqslant k \leqslant m-1}$, has a unique solution given by

$$C_{k,j}(\xi) = \sum_{l=0}^{m-1} P_{k,l}(\xi)\, \delta_{l,j} = P_{k,j}(\xi),$$

where $\| P_{k,j}(\xi) \|$ is the inverse matrix of $\| p_{j,k}(\xi) \|$.

Then, we have the following result. *Under the above assumptions the function*

$$U_j(\xi, t) = \sum_{k=0}^{m-1} P_{k,j}(\xi) \int_{C_\xi} \frac{\lambda^k e^{i\lambda t}}{a(\xi, \lambda)}\, d\lambda \tag{3.5}$$

is, for each $0 \leqslant j \leqslant m - 1$, *a solution of problem* (3.3).

Next, let us examine how the homogeneity of the operators A and B_j reflect into the function $U_j(\xi, t)$. Consider each term

$$U_{k,j}(\xi, t) = P_{k,j}(\xi) \int_{C_\xi} \frac{\lambda^k e^{i\lambda t}}{a(\xi, \lambda)}\, d\lambda \tag{3.6}$$

appearing in the expression (3.5).

First of all, we observe that $p_{j,k}(\xi)$ defined by (3.1) is a homogeneous function of degree $m_j + k + 1 - 2m$ in ξ. In fact, by changing variables in (3.1) we get:

$$p_{j,\xi}(\xi) = |\xi|^{m_j+k+1-2m} \int_C \frac{\mu^k b_j\left(\frac{\xi}{|\xi|}, \mu\right)}{a\left(\frac{\xi}{|\xi|}, \mu\right)} d\mu.$$

Secondly, we have:

$$P_{k,j}(\xi) = \Delta_{j,k}/\Delta$$

where $\Delta_{j,k}$ is the determinant of the co-factor of $p_{j,k}(\xi)$ and Δ is the determinant of $\| p_{j,k}(\xi)\|$. It is easy to see that Δ is homogeneous of degree

$$(m_0 + \cdots + m_{m-1}) + \frac{m(m-1)}{2} + (1-2m)(m-1)$$

while $\Delta_{j,k}$ is homogeneous of degree

$$(m_0 + \cdots + \hat{m}_j + \cdots + m_{m-1}) + (1 + \cdots + \hat{k} + \cdots + (m-1)) + (1-2m)(m-2),$$

where $\hat{}$ indicates that the number below it is delated. The degree of homogeneity of $P_{k,j}(\xi)$ is then the difference between the degree of $\Delta_{j,k}$ and the degree of Δ, i.e., equal to $2m - m_j - k - 1$.

Let, now, $\xi = r\omega$ where $r = |\xi|$ and $\omega = (\omega_1, \ldots, \omega_n)$ with $|\omega| = 1$. By changing variables in (3.6) and by taking into account the degree of homogeneity of $P_{k,j}(\xi)$ and that of $a(\xi, \lambda)$ we can write:

$$U_{k,j}(\xi, t) = r^{-m_j} f_{k,j}(r, \omega, t) \tag{3.7}$$

where

$$f_{k,j}(r, \omega, t) = P_{k,j}(\omega) \int_C \frac{\mu^k e^{ir\mu t}}{a(\omega, \mu)} d\mu. \tag{3.8}$$

Thus, for each $0 \leqslant j \leqslant m-1$, we have:

$$U_j(\xi, \mathrm{t}) = r^{-m_j} f_j(r, \omega, t) \tag{3.9}$$

where

$$f_j(r, \omega, t) = \sum_{k=0}^{m-1} f_{k,j}(r, \omega, t). \tag{3.10}$$

Since $f_j(r, \omega, t)$ is a smooth function we can apply the results of Sections 1 and 2. First of all, $U_j(\xi, t)$ defines a tempered distribution in R^n, depending upon the parameter $t > 0$.

For each $0 \leqslant j \leqslant m - 1$, let

$$K_j(x, t) = \mathcal{F}_\xi^{-1}(U_j(\xi, t)). \tag{3.11}$$

From Theorem 1, it follows that, for each $t > 0$, $K_j(x, t)$ is a C^∞ function in $R^n - \{0\}$. Furthermore, when $m_j < n$, $K_j(x, t)$ is a homogeneous function of degree $m_j - n$ in the variable x while, when $m_j \geqslant n$, $K_j(x, t)$ decomposes as (2.1) in the variable x (Theorem 3).

In fact, we shall see that $K_j(x, t)$ is C^∞ in R_+^{n+1} and it verifies homogeneity properties in (x, t).

4. Elementary Solution of the Elliptic Boundary Problem

Consider the following function

$$U_{k,j}(\xi) = r^{-m_j} f_{k,j}(\omega) \tag{4.1}$$

where

$$f_{k,j}(\omega) = P_{k,j}(\omega) \int_C \frac{\mu^k}{a(\omega, \mu)}\, d\mu \tag{4.2}$$

(i.e. $U_{k,j}(\xi) = U_{k,j}(\xi, 0)$).

As before, $U_{k,j}(\xi)$ defines a tempered distribution in R^n. We want to prove the following

THEOREM 4. *When $t \to 0+$, $U_{k,j}(\xi, t)$ converges to $U_{k,j}(\xi)$ in $\mathcal{S}'(R^n)$.*

The proof is based upon the following

LEMMA. *For every integer $p > 0$,*

$$r^p u_\phi(r, t) \to r^p u_\phi(r)$$

when $t \to 0+$, uniformly on $r \in [0, \infty)$ and on $\phi \in B$, where B is a bounded set in $\mathcal{S}(R^n)$.

Proof. Let

$$r^p(u_\phi(r, t) - u_\phi(r)) = \int_\Omega \left\{ P_{k,j}(\omega) \int_C \frac{\mu^k(e^{ir\mu t} - 1)}{a(\omega, \mu)}\, d\mu \right\} r^p \phi(r\omega)\, d\omega.$$

Given $\epsilon > 0$, it is possible to find (because $\phi \in \mathcal{S}$) a constant $A > 0$, sufficiently large, such that

$$\left| \int_\Omega \left\{ P_{k,j}(\omega) \int_C \frac{\mu^k e^{ir\mu t}}{a(\omega, \mu)} \, d\mu \right\} r^p \phi(r\omega) \, d\omega \right| < \frac{\epsilon}{3},$$

$\forall r > A$, $\forall t > 0$, and $\forall \phi$ in a fixed bounded set B of $\mathcal{S}$, and such that

$$\left| \int_\Omega \left\{ P_{k,j}(\omega) \int_C \frac{\mu^k}{a(\omega, \mu)} \, d\mu \right\} r^p \phi(r\omega) \, d\omega \right| < \frac{\epsilon}{3},$$

$\forall r > A$ and $\forall \phi \in B$.

Next, since $e^{ir\mu t} - 1 \to 0$ uniformly when $r \leqslant A$ and $\mu \in C$, as $t \to 0^+$, then we can find $\delta > 0$, such that

$$\left| \int_\Omega \left\{ P_{k,j}(\omega) \int_C \frac{\mu^k (e^{ir\mu t} - 1)}{a(\omega, \mu)} \, d\mu \right\} r^p \phi(r\omega) \, d\omega \right| < \frac{\epsilon}{3}$$

uniformly when $r \leqslant A$ and $\phi \in B$, whenever $|t| < \delta$.

Combining these three inequalities we get

$$| r^p (u_\phi(r, t) - u_\phi(r))| < \epsilon,$$

$\forall r \in [0, \infty)$ and $\forall \phi \in B$, whenever $|t| < \delta$, Q.E.D.

Proof of Theorem 4. We shall consider only the case $m_j > n - 1$. Case $m_j = m - 1$ is similar and case $m_j < n - 1$ is easier.

Let B be a bounded set of $\mathcal{S}(R^{n-1})$, let $\phi \in B$ and consider expression (3.7). By taking derivative of $u_\phi(r, t)$ given by (3.4), it is not difficult to check that, when $t \to 0^+$,

$$\frac{\partial s}{\partial_r s} u_\phi(0, t) \to u_\phi^{(s)}(0)$$

uniformly when $\phi \in B$.

Thus, using the lemma, the term between brackets in the first integral appearing in (3.7) converges, when $t \to 0^+$, to

$$u_\phi(r) - \sum_{r=0}^{m_j - n + 1} \frac{1}{s!} u_\phi^{(s)}(0) \cdot r^s$$

uniformly on $r \in [0, 1]$ and $\phi \in B$. Then we can take limits inside the integral sign.

Again, using the lemma, it is easy to see that

$$\int_1^\infty r^{-m_j+n-2} u_\phi(r, t)\, dr \to \int_1^\infty r^{m_j+n-2} u_\phi(r)\, dr,$$

when $t \to 0^+$, $\forall \phi \in B$. Finally,

$$\frac{\partial^s u_\phi}{\partial_r s}(0, t) \int_1^\infty r^{-m_j+n-2+s}\, dr \to u_\phi^{(s)}(0) \int_1^\infty r^{-m_j+n-2+s}\, dr,$$

when $t \to 0^+$, $\forall \phi \in B$, by virtue of our assumption on m_j.

From these results, it follows that

$$\langle U_{k,j}(\xi, t), \phi(\xi)\rangle \to \langle U_{k,j}(\xi), \phi(\xi)\rangle$$

as $t \to 0^+$, uniformly when ϕ belongs to a bounded set of $\mathcal{S}(R^{n-1})$, i.e.

$$U_{k,j}(\xi, t) \to U_{k,j}(\xi) \quad \text{in} \quad \mathcal{S}'(R^{n-1}) \qquad \text{Q.E.D.}$$

From Theorem 4, we derive the following

COROLLARY. *For each* $0 \leqslant j \leqslant m - 1$, K_j *is a solution of problem* (3.2).

Proof. As we already remarked, $U_j(\xi, t)$ is a solution of problem (3.3). By taking inverse Fourier transforms, we have trivially

$$AK_j = 0 \quad \text{in} \quad R_+^{n+1}.$$

Since $U_j(\xi, t) \to U_j(\xi)$, again by taking inverse Fourier transforms we get

$$\lim_{t \to 0+} B_l K_j(x_1, ..., x_n, t) = \delta_{l,j} \cdot \delta,$$

this limit being taken in $\mathcal{S}'(R_+^{n+1})$, Q.E.D.

Another consequence of Theorem 4 is that we can extend the definition of $K_j(x, t)$ when $t = 0$. It suffices to put

$$K_j(x, 0) = \mathcal{F}_\xi^{-1} U_j(\xi).$$

5. A Representation Formula for the Kernels.

As we remarked in Section 1, it is not always possible to represent the Fourier transform $\mathcal{F}U$, considered there, by means of absolutely convergent integrals. When $m > n$, the representation is always possible; when $m \leqslant n$, we can only represent the term near the origin (see 1.11 and 1.12).

However, in the cases under consideration, the distributions K_j, $0 \leqslant j \leqslant m-1$, can always be represented by absolutely convergent integrals. It is the presence of the exponential term in (3.8), where $t > 0$ and $\mu \in C$ belongs to the complex upper half plane, that makes such a representation possible. We have the following

THEOREM 5. *Let $0 \leqslant j \leqslant m-1$. If j is such that:*

1) $m_j < n$, *then*

$$K_j(x, t) = (2\pi)^{-n} \int_{R^n} U_j(\xi, t)\, e^{i\langle x, \xi\rangle}\, d\xi; \tag{5.1}$$

2) $m_j = n$, *then*

$$K_j(x, t) = (2\pi)^{-n} \int_{R^n} \mathfrak{X}(\xi) \cdot U_j(\xi, t)(e^{i\langle x,\xi\rangle} - 1)\, d\xi$$

$$+ (2\pi)^{-n} \int_{R^n} (1 - \mathfrak{X}(\xi))\, U_j(\xi, t)\, e^{i\langle x,\xi\rangle}\, d\xi; \tag{5.2}$$

3) $m_j > n$, *then*

$$K_j(x, t) = (2\pi)^{-n} \int_{R^n} \mathfrak{X}(\xi)\, U_j(\xi, t) \left(e^{i\langle x,\xi\rangle} - \sum_{s=0}^{m_j-n} \frac{1}{s!} i\langle x, \xi\rangle\right)^s d\xi$$

$$+ (2\pi)^{-n} \int_{R^n} (1 - \mathfrak{X}(\xi))\, U_j(\xi, t) \left(e^{i\langle x,\xi\rangle} \sum_{s=0}^{m_j-n-1} \frac{1}{s!} i\langle x, \xi\rangle\right)^s d\xi; \tag{5.3}$$

all the integrals being absolutely convergent

Proof. Formula (5.3) is a direct consequence of (1.6).

To prove (5.1), consider the following integral

$$\int_C \int_{R^n} \frac{P_{k,j}\left(\frac{\xi}{|\xi|}\right)\mu^k}{a\left(\frac{\xi}{|\xi|}, \mu\right)} \frac{e^{i(\langle x,\xi\rangle + |\xi|\mu t)}}{|\xi|^{m_j}}\, d\mu\, d\xi. \tag{5.4}$$

Since

$$\int_{R^n} \frac{e^{i(\langle x,\xi\rangle + |\xi|\mu t)}}{|\xi|^{m_j}}\, d\xi \tag{5.5}$$

is absolutely convergent, because $t > 0$, $\operatorname{Im} \mu > 0$ and $m_j < n$, then (5.4) is also absolutely convergent. By Fubini's theorem and by changing variables we see that (5.4) is equal to the integral

$$\int_{R^n} U_{k,j}(\xi, t)\, e^{i\langle x,\xi\rangle}\, d\xi.$$

Hence (5.1) is absolutely convergent.

The proof that the second integral in (5.2) is absolutely convergent follows the same pattern as the previous one. We already know (see (1.11)) that the first integral appearing in (5.2) is absolutely convergent. This completes the proof. Q.E.D.

Remark. Formulas (5.1), (5.2) and (5.3) show that $K_j(x, t)$ is well defined when $x = 0$, for all $t > 0$. This, together with the observation following the proof of the corollary of Theorem 4, shows that $K_j(x, t)$ is defined in $\overline{R^{n+1}_+} - \{0\}$, for all j.

Next, by defining

$$K_j(x, t) = K_j(x, -t)$$

when $t < 0$, we can extend $K_j(x, t)$ to $R^{n+1} - \{0\}$.

Going back to the definition of $K_j(x, t)$, we observe that, for each $t > 0$, $K_j \in \mathcal{S}'(R^{n+1}_+)$. This because U_j is itself an element of $\mathcal{S}'(R^{n+1}_+)$.

Furthermore, an easy verification show us that, when $m_j < n$, K_j is a homogeneous distribution of degree $m_j - n$ in (x, t). If $m_j \geqslant n$, the same computation as in Theorem 3, gives the decomposition of K_j as the sum of a homogeneous function of degree $m_j - n$ in (x, t), plus a homogeneous polynomial of degree $m_j - n$ in (x, t), times a logarithm.

As we remarked, already, $K_j(x, t)$ is C^∞ in $R^n - \{0\}$ for each $t > 0$. It is easy to see, using formulas (5.1), (5.2) and (5.3) that $K_j(x, t)$ is C^∞ in R^{n+1}_+.

Finally, if we want to consider $K_j(x, t)$ extended to $R^{n+1}_+ - \{0\}$, again the representation formulas, together with the definition of $K_j(x, 0)$, show that $K_j(x, t)$ is C^∞ in $R^{n+1} - \{0\}$.

We can summarize these results in the following

THEOREM 6. *Let $(A, (B_j)_{0 \leqslant j \leqslant m-1})$ be a regular elliptic problem in R^{n+1}_+. There are m temperated distributions $K_j \in \mathcal{S}(R^{n+1})$, $0 \leqslant j \leqslant m-1$, such that:*

a) *each K_j is a solution of problem* (3.2);

b) *each K_j is a C^∞ function in $R^{n+1} - \{0\}$*;

c) *if* $m_j < n$, *then* K_j *is homogeneous of degree* $m_j - n$;

d) *if* $m_j \geqslant n$, *then* $K_j = G_j + P_j \cdot \ln(x_1^2 + \cdots + x_n^2 + t^2)^{1/2}$,

where G_j *is a homogeneous function of degree* $m_j - n$ *and* P_j *is a homogeneous polynomial of degree* $m_j - n$.

A temperated distribution verifying properties b), c) and d) of Theorem 6 was called, by Seeley [7], a *modified* C^∞ *homogeneous distribution of degree* $m_j - n$.

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